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Analytical study on the motion of a system with varying damping subject to harmonic force

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Abstract

To show the possibility of a new kind of a moment shaker, the motion of a system composed of a rigid plate, constant springs and time-varying dampers has been considered when the plate is subjected to harmonic force. The vertical displacement at the center of the plate and the rotational displacement of the plate were obtained by numerical integration of the equations of motion. The major frequency components of these responses were found through fast Fourier transform and the condition under which the plate oscillates with almost a single frequency and a large amplitude was found. The existence of the frequency components of the vertical and rotational displacements of the plate has been proved analytically. Approximate expressions for the amplitude of the vertical and rotational displacements have been derived for a special case.
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1. Introduction

The responses of dynamic systems with time-varying (usually periodic) parameters have been studied by many researchers. This kind of response occurs in structural systems subject to turbulent flow, cracked rotors, elastic linkage systems such as slider-crank mechanisms, and drive trains where the inertias vary periodically because of the radial motion of the pistons. Systems governed by the Mathieu equation are common examples of dynamic systems with periodic parameters, and they are described in textbooks [1]. When an external force is periodic, the computation of the steady-state response is required. A large amount of research has been published on the development of methods to determine the stability of the responses and methods to compute the steady-state responses. The commonly used procedure to find analytically the responses of the systems with time-varying parameters consists of using the methods based on Fourier series expansion [2]. Additionally, a modal analysis method was developed that predicts the steady-state response of discrete linear systems with periodic parameters [3]. In the method, the solution for the steady-state response is expressed as a linear combination of the Floquet eigenvectors, which are orthogonal with respect to solutions of the

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associated adjoint problem. An approximate closed-form solution for a linear slowly varying system under external excitation was derived based on the technique of freezing slowly varying parameters [4].

Dynamic systems with periodic parameters, which have been considered, may be grouped from the viewpoint of the fundamental frequency of the external force. The case where the fundamental frequency of the external force is equal to the one of the periodic parameters is treated in Ref. [3], the case where the frequencies of the external force and of periodic parameters are distinct in Refs. [2,5]. Another viewpoint is which parameters of dynamic systems vary periodically. Most analyses concern the case where damping coefficients are constant, while the stiffness is periodically time varying. Some other analyses concern the case of synchronous stiffness and damping variation, which can be applied to suppress self-excited vibrations [6]. In this paper, a rigid plate with two constant springs and two dampers with periodic damping coefficients is considered. The plate is subject to a harmonic force whose frequency is different from the one of damping coefficient variation. The current system was devised during the development process of a shaker of new type explained below.

To investigate experimentally the dynamic behavior of large structures with low natural frequencies such as buildings, towers, and bridges, shakers, which can deliver low-frequency excitation forces to these structures, are needed. Shakers using out-of-balance masses [7,8] and servo-hydraulic shakers [9] have been mainly utilized for this purpose.

For supplying low-frequency excitation forces, a new type of shaker, which utilizes force frequency shifting, was proposed [10–12]. The idea of the new shaker is as follows. If an excitation force $F = F_0 \sin \omega t$ applies on a structure, and the force application point moves back and forth along the structure with $s = s_0 + r \sin \omega_2 t$ as shown in Fig. 1, generalized forces with frequencies $\omega_1 - \omega_2$ and $\omega_1 + \omega_2$ are generated and excite the structure. Therefore, by moving an ordinary out-of-balance mass exciter along a structure and adjusting the two frequencies, ω_1 and ω_2, one can get an excitation force with a desired low frequency. However, this method is inconvenient because of the requirement to move an exciter back and forth normal to excitation force along a structure.

A method has been proposed which provides force frequency shifting without moving an exciter and is shown in Fig. 2 [13]. This excitation system is composed of a plate, springs and dampers. An excitation force with frequency ω_1 applies from an out-of-balance mass exciter to the center of the plate, and only one pair of a spring and a damper is active at any instant. That is, this pair of a spring and a damper has finite spring or damping constants and the remaining pairs have zero spring or damping constants. If active springs and

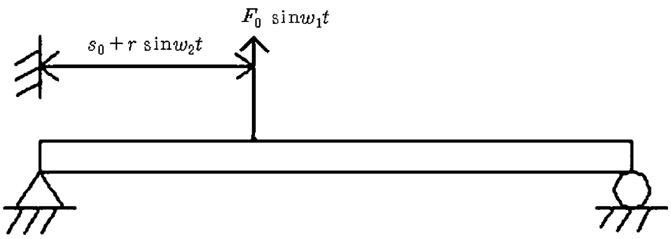

Fig. 1. Force frequency shifting with a reciprocating shaker.

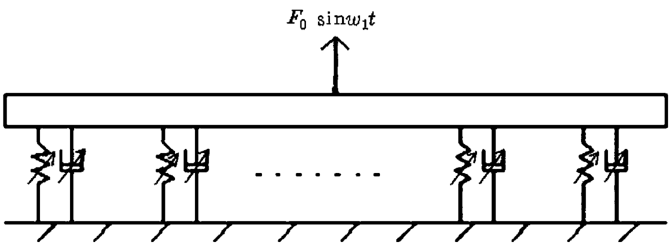

Fig. 2. A shaker composed of a plate, springs, and dampers.

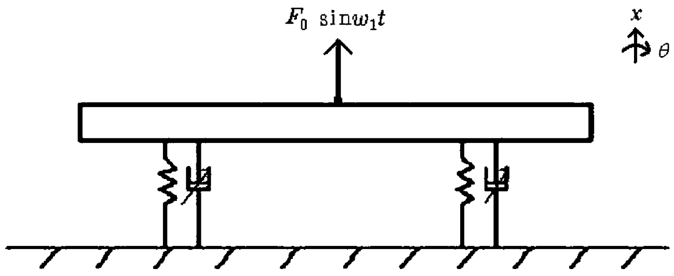

Fig. 3. A two-spring–damper system.

dampers vary with frequency ω_2 along the plate, spring forces and damping forces will move with the frequency along the plate. It was found in Ref. [13] that the structure which the excitation system is attached to was acted upon a generalized force with the difference frequency $\omega_1 - \omega_2$.

To analyze the dynamic behavior of this excitation system, a simple system with two springs and dampers shown in Fig. 3 is considered in this paper. The considered system is symmetric in configuration. The spring constants of the two springs are the same and constant with time, and the damping constants of the dampers vary with time. The motion of the plate under these conditions is investigated through numerical calculations. It is tried to explain analytically the existence of major frequency components of the plate's motion, which are found through numerical calculation.

2. Numerical analysis of the motion

To analyze the motion of the system shown in Fig. 3 easily, the plate is assumed to be rigid. The equation of motion of the plate is derived as follows:

$$m\ddot{x} + (c_1 + c_2)\dot{x} + (c_1 - c_2)r\dot{\theta} + 2kx = F_0 \sin \omega_1 t, \tag{1}$$

$$J\ddot{\theta} + (c_1 - c_2)r\dot{x} + (c_1 + c_2)r^2\dot{\theta} + 2kr^2\theta = 0, \tag{2}$$

where m, J, x, θ, F_0, k, c_i, and r represent the mass and the mass moment of inertia of the plate, the vertical displacement at the center of the plate and the rotational displacement of the plate, the amplitude of the excitation force, the spring constant, the damping constant of each damper, and the distance to each spring and damper from the center of the plate, respectively. It is assumed that a pair of a spring and a damper is attached to the same point on the plate. The damping constants of the considered system were made to vary as square waves. The variation of damping constants in Fig. 4 shows that two dampers become active alternately.

If dampers are deleted from the above system, the equations of motion become decoupled and two natural frequencies become the natural frequency of the translational mode, ω_{nd}, and that of the rotational mode, ω_{nr}. These natural frequencies are obtained as follows:

$$\omega_{\mathrm{nd}} = \sqrt{\frac{2k}{m}}, \tag{3}$$

$$\omega_{\mathrm{nr}} = \sqrt{\frac{2kr^2}{J}}. \tag{4}$$

The motion of the system was analyzed by numerical integration for a homogeneous case where the external excitation was removed from Eq. (1) at first. The equations of motion of the system, Eqs. (1) and (2), were solved for given system parameters using MATLAB. The system parameters were chosen to be $m = 30\,\mathrm{kg}$, $J = 100\,\mathrm{kg\,m^2}$, $k = 3.5 \times 10^4\,\mathrm{N/m}$, $(c_1)_{\max} = (c_2)_{\max} = 1 \times 10^3\,\mathrm{N\,s/m}$, $r = 1/3\,\mathrm{m}$, and the damping constants were made to vary according to Fig. 4 with frequency $f_2 = \omega_2/2\pi = 9\,\mathrm{Hz}$. f_2 represents the on–off frequency of a damper, and if $f_2 = 10\,\mathrm{Hz}$, it means that dampers become on and off 10 times per second. In this paper,

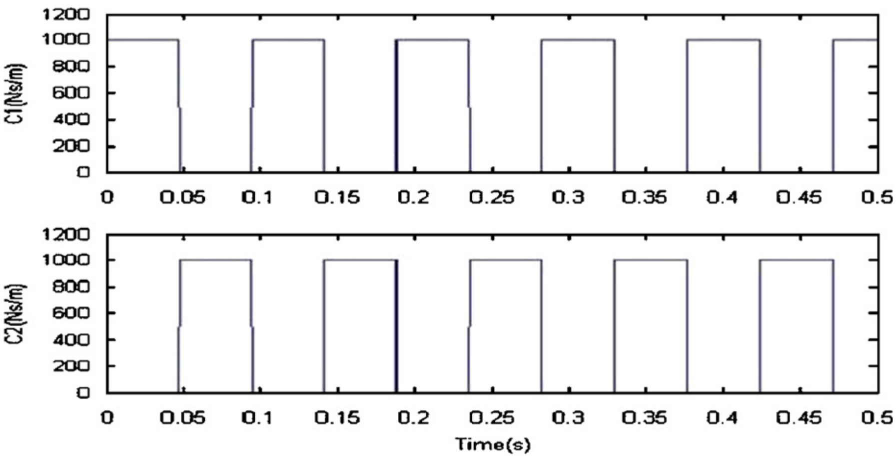

Fig. 4. Variation of the damping constants for a two-spring–damper system.

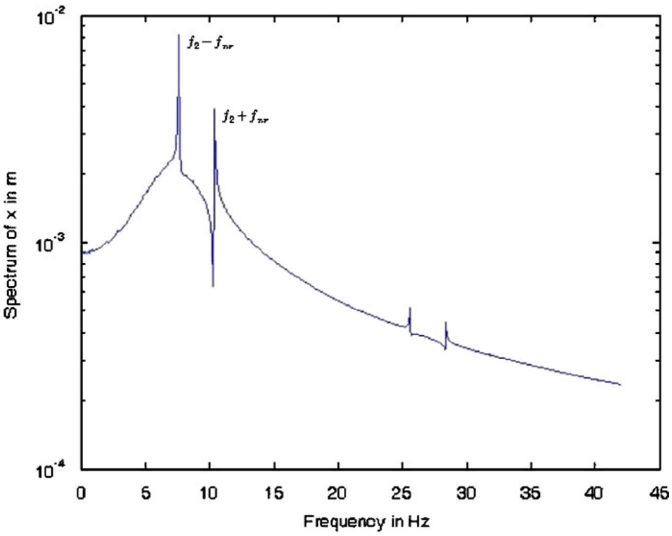

Fig. 5. Frequency spectrum of the vertical displacement at the center of the plate for a homogeneous case.

two symbols for frequency, f and ω, are used together. f is in units of Hz and ω in units of rad/s. This set of parameters correspond to the natural frequencies of the translational and rotational modes, $f_{nd} = \omega_{nd}/2\pi = 7.69$ Hz and $f_{nr} = \omega_{nr}/2\pi = 1.40$ Hz. The initial conditions for $x, \dot{x}, \theta$, and $\dot{\theta}$ were all set to unity. The time interval between calculated data points was $0.005/12 = 0.4167 \times 10^{-3}$ s and the length was 80,000 data points. Of course, the vertical displacement x and the rotational displacement θ of the plate diminished with time due to damping. The vertical and rotational displacements reduced to 1% of their initial values after 9.4 and 23.6 s, respectively. The obtained time data of x and θ were Fourier transformed to calculate their frequency components. The frequency interval between transformed data became 0.03 Hz. Figs. 5 and 6 show frequency components of x and θ. The frequency components of x show peaks at 7.62 and 10.41 Hz with negligible other components, and these frequencies correspond to f_2–f_{nr} and f_2+f_{nr}, respectively. The frequency components of θ show a single peak at 1.38 Hz, corresponding to f_{nr}.

Next the case with external excitation was considered. The responses of the system with the same parameters as above were obtained for $f_1 = \omega_1/2\pi = 12$ Hz and $F_0 = 1 \times 10^3$ N. The initial conditions for $x, \dot{x}, \theta$, and $\dot{\theta}$ were zeros. 160,000 data points with the time interval of $0.005/f_1 = 0.4167 \times 10^{-3}$ s were obtained and the first 80,000 data points were deleted so that the transient response would be damped out. The 80,000 data points

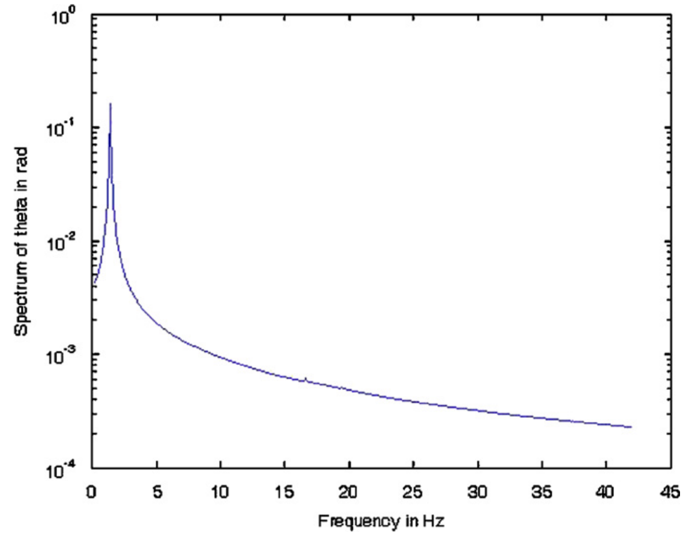

Fig. 6. Frequency spectrum of the rotational displacement of the plate for a homogeneous case.

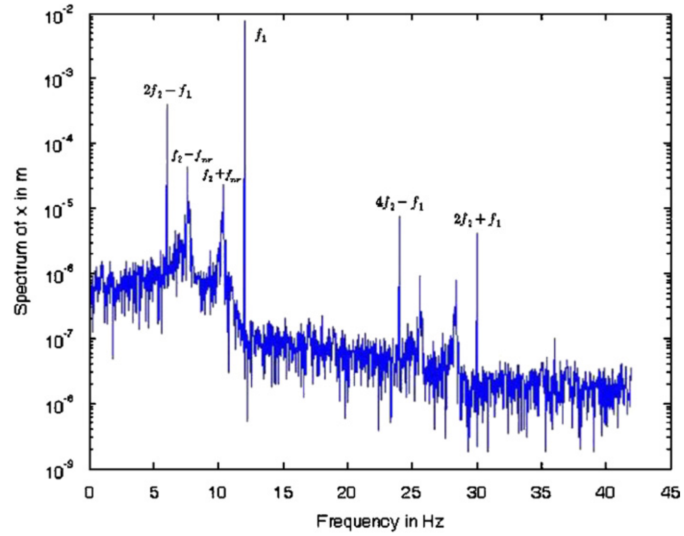

Fig. 7. Frequency spectrum of the vertical displacement at the center of the plate for a case with external excitation.

correspond to the period of 33.3 s, which is long enough for the transient response to be damped out based on the result of the homogeneous case. Figs. 7 and 8 show frequency components of x and θ. The frequency components of x show peaks at 6, 7.62, 10.38, 12, 24, and 30 Hz, and these frequencies correspond to $2f_2-f_1$, $f_2-f_{\mathrm{nr}}, f_2+f_{\mathrm{nr}}, f_1$, $4f_2-f_1$, and $2f_2+f_1$, respectively. Among these frequency components, the component at f_1 is the strongest and the one at $2f_2-f_1$ is the next. The frequency components of θ show peaks at 1.38, 3, 15, 21, 33, and 39 Hz, and these frequencies correspond to f_{nr}, f_1-f_2, $3f_2-f_1$, f_1+f_2, $5f_2-f_1$, and $3f_2+f_1$, respectively. Among these frequency components, the components at f_{nr} and f_1-f_2 are prominent. It should be noticed that the frequency components of transient responses for the homogeneous case appear also in the responses with transient parts removed.

Calculating the frequency components of x and θ for different values of f_2, the same phenomena as above were observed. For a special case where f_1-f_2 coincides with the natural frequency of the rotational mode,

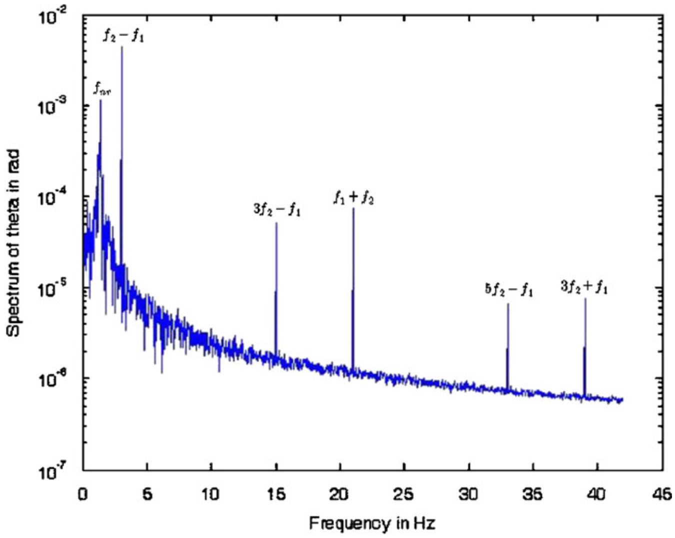

Fig. 8. Frequency spectrum of the rotational displacement of the plate for a case with external excitation.

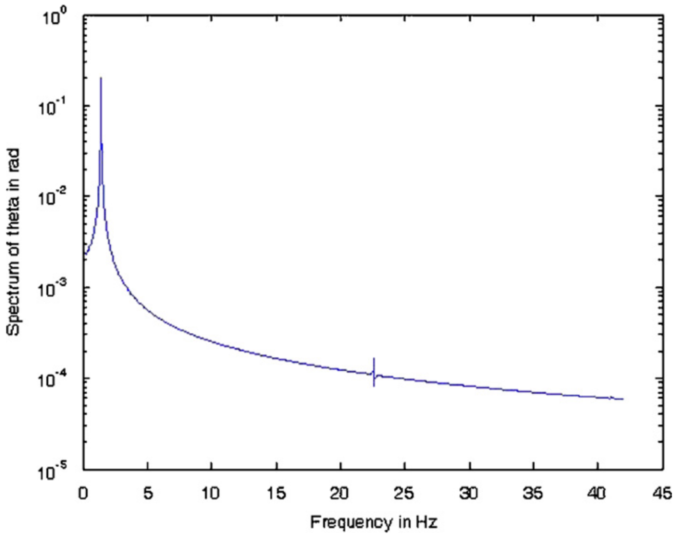

Fig. 9. Frequency spectrum of the rotational displacement when the difference frequency coincides with the natural frequency of the rotational mode.

f_{nr}, two peaks for these frequencies merge and result in one prominent peak with negligible peaks at other frequencies as Fig. 9 shows. This observation means that θ has almost a single frequency component for this case.

The maximum frequency components of θ were calculated for f_1 equal to 12 Hz and various values of f_2, and are shown in Fig. 10. In the figure, the horizontal axis represents the ratio of $(f_1-f_2)/f_{nr}$, and the vertical axis the ratio of the maximum frequency component to the global maximum value. These maximum components occurred at f_{nr} or f_1-f_2 for each value of f_2. The maximum frequency component of θ increases rapidly around the frequency where f_1-f_2 coincides with f_{nr}, that is, $(f_1-f_2)/f_{nr}$ is equal to 1. Similarly, when f_2 is greater than f_1, the same result was observed. The maximum frequency component of θ increases rapidly when $(f_1-f_2)/f_{nr}$ is equal to -1. This result implies that the plate of the system can be made to oscillate with almost a single

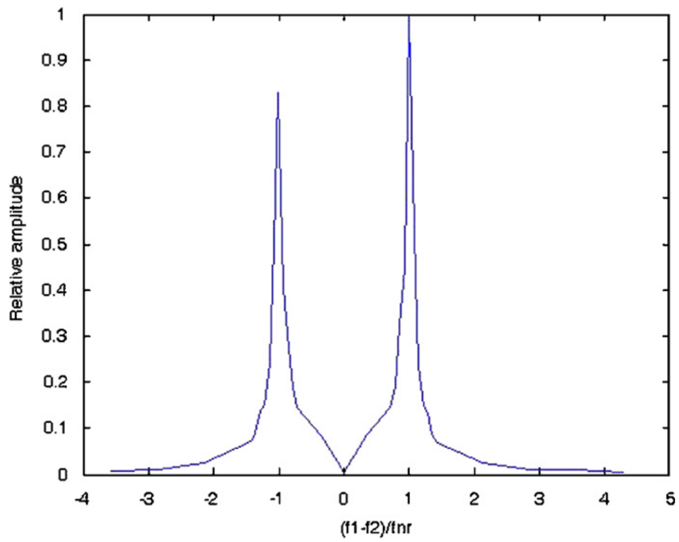

Fig. 10. Variation of the maximum frequency component of the rotational displacement with frequency f_2.

frequency and a large amplitude by adjusting frequency f_2 so that $|f_1{-}f_2|$ coincides with f_{nr}. If this system is mounted on a structure, this oscillatory motion of the plate will deliver a moment with frequency $|f_1{-}f_2|$ to the structure. Therefore, this system works as a moment shaker. When f_2 is equal to $f_1 = 12\,\mathrm{Hz}$, the amplitude of θ decreases with time and finally converges to zero. Consequently, the maximum component of θ for $f_2 = 12\,\mathrm{Hz}$ becomes very small.

3. Analytical analysis

3.1. Proof of frequency components in x and θ

The frequency components of vertical and rotational displacements of the plate were identified by numerical calculation in the previous section. It has been tried to prove the results analytically in this section.

The equations of motion of the considered system are given in Eqs. (1) and (2). If c_1 and c_2 vary with time in a manner shown in Fig. 4, and their maximum values are set to c, $c_{1+}c_2$ becomes a constant equal to c, and $c_1{-}c_2$ becomes a square wave with amplitude c. Expressing the square wave as a Fourier series, the equations of motion become

$$m\ddot{x} + c\dot{x} + cr\left(\frac{4}{\pi}\sin\,\omega_2 t + \frac{4}{3\pi}\sin\,3\omega_2 t + \cdots\right)\dot{\theta} + 2kx = F_0\,\sin\,\omega_1 t, \tag{5}$$

$$J\ddot{\theta} + cr\left(\frac{4}{\pi}\sin\,\omega_2 t + \frac{4}{3\pi}\sin\,3\omega_2 t + \cdots\right)\dot{x} + cr^2\dot{\theta} + 2kr^2\theta = 0. \tag{6}$$

Since the plate of the system is subject to a force $F_0 \sin \omega_1 t$, it is easily expected that the vertical displacement at the center of the plate, x, has a frequency component at ω_1. Therefore, including only one frequency component at ω_1 in x,

$$x = a_1\,\sin\,\omega_1 t + b_1\,\cos\,\omega_1 t \tag{7}$$

and substituting the equation in Eq. (6) and using the trigonometric relations

$$\sin\,\alpha\,\sin\,\beta = \tfrac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)], \tag{8}$$

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)], \tag{9}$$

the term containing $\dot{x}$ in Eq. (6) becomes as follows:

$$\begin{aligned}\left(\frac{4}{\pi}\sin\omega_2 t + \frac{4}{3\pi}\sin 3\omega_2 t + \cdots\right)\dot{x} = {} & \frac{2}{\pi}a_1\omega_1[\sin(\omega_1+\omega_2)t - \sin(\omega_1-\omega_2)t] \\ & - \frac{2}{\pi}b_1\omega_1[\cos(\omega_1-\omega_2)t - \cos(\omega_1+\omega_2)t] \\ & + \frac{2}{3\pi}a_1\omega_1[\sin(3\omega_2+\omega_1)t + \sin(3\omega_2-\omega_1)t] \\ & - \frac{2}{3\pi}b_1\omega_1[\cos(3\omega_2-\omega_1)t - \cos(3\omega_2+\omega_1)t] + \cdots . \end{aligned} \tag{10}$$

Since the term containing $\dot{x}$ in Eq. (6) produces frequency components at $\omega_1-\omega_2$, $\omega_1+\omega_2$, $3\omega_2-\omega_1$, $3\omega_2+\omega_1$, and so on, θ must have these frequency components also to cancel out these components in Eq. (6).

From the above results, one can let

$$\theta = d_1\sin(\omega_1+\omega_2)t + e_1\cos(\omega_1+\omega_2)t + d_2\sin(\omega_1-\omega_2)t + e_2\cos(\omega_1-\omega_2)t + \cdots . \tag{11}$$

Substituting the above equation in Eq. (5), one can find that the term containing $\dot{\theta}$ possesses frequency components at ω_1, $\omega_1+2\omega_2$, $2\omega_2-\omega_1$, $\omega_1+4\omega_2$, $4\omega_2-\omega_1$, and so on. In order to cancel out these frequency components in Eq. (5), x must possess these frequency components also.

The frequency component of θ at $\omega_{nr} = \sqrt{2kr^2/J}$ is a solution of the equation $J\ddot{\theta} + 2kr^2\theta = 0$, which is a part of Eq. (6). If one can assume that the remaining terms in Eq. (6) have ω_{nr} components with negligible magnitudes compared to $J\ddot{\theta}$ and $2kr^2\theta$ terms, ω_{nr} component can be a component of θ. Even though the above assumption was found to be reasonable through numerical calculation, this procedure of proof is not satisfactory. However, the authors cannot find a better procedure. If it is proved that θ possesses a frequency component at ω_{nr}, one can prove that x possesses frequency components at $\omega_2-\omega_{nr}$ and $\omega_2+\omega_{nr}$ by substituting θ with ω_{nr} component into Eq. (5) and following the similar procedure as in the above paragraph. Now, the proof of existence of major frequency components of x and θ has been completed.

3.2. Expressions of x and θ

It was found through numerical analysis that the rotational displacement θ has a prominent component at $\omega_1-\omega_2$ with negligible other components when this frequency coincides with the natural frequency of the rotational mode, ω_{nr}, that is, $\omega_1-\omega_2=\omega_{nr}=\sqrt{2kr^2/J}$. For this reason, approximate expressions of x and θ for this case have been derived in this section. Only one frequency component at $\omega_1-\omega_2$ was included in θ, and two components at ω_1 and $2\omega_2-\omega_1$ in x. Of course more frequency components might be included in the assumed response to improve the accuracy of the solution. However, it would result in more complicated expressions for x and θ.

Including the above-mentioned components, x and θ can be written as follows:

$$x = a_1\sin\omega_1 t + b_1\cos\omega_1 t + a_2\sin(2\omega_2-\omega_1)t + b_2\cos(2\omega_2-\omega_1)t, \tag{12}$$

$$\theta = d_1\sin(\omega_1-\omega_2)t + e_1\cos(\omega_1-\omega_2)t. \tag{13}$$

Substituting the above equations to Eqs. (5) and (6), collecting sine and cosine terms with the same frequencies, and comparing the coefficients of the terms with $\sin\omega_1 t$, $\cos\omega_1 t$, $\sin(2\omega_2-\omega_1)t$, $\cos(2\omega_2-\omega_1)t$, $\sin(\omega_1-\omega_2)t$, and $\cos(\omega_1-\omega_2)t$, one can obtain the following set of equations:

$$(2k-m\omega_1^2)a_1 - c\omega_1 b_1 + \frac{2}{\pi}cr(\omega_1-\omega_2)d_1 = F_0, \tag{14}$$

$$c\omega_1 a_1 + (2k-m\omega_1^2)b_1 + \frac{2}{\pi}cr(\omega_1-\omega_2)e_1 = 0, \tag{15}$$

$$[2k - m(2\omega_2 - \omega_1)^2]a_2 - c(2\omega_2 - \omega_1)b_2 + \frac{2}{\pi}cr(\omega_1 - \omega_2)d_1 = 0, \tag{16}$$

$$c(2\omega_2 - \omega_1)a_2 + [2k - m(2\omega_2 - \omega_1)^2]b_2 - \frac{2}{\pi}cr(\omega_1 - \omega_2)e_1 = 0, \tag{17}$$

$$-\frac{2}{\pi}cr\omega_1 a_1 + \frac{2}{\pi}cr(2\omega_2 - \omega_1)a_2 - cr^2(\omega_1 - \omega_2)e_1 = 0, \tag{18}$$

$$-\frac{2}{\pi}cr\omega_1 b_1 - \frac{2}{\pi}cr(2\omega_2 - \omega_1)b_2 + cr^2(\omega_1 - \omega_2)d_1 = 0. \tag{19}$$

Note that J, the mass moment of inertia, is not shown in the above equations because of the imposed condition, $\omega_1 - \omega_2 = \sqrt{2kr^2/J}$.

The solutions of the above simultaneous equations for the coefficients, a_1, b_1, a_2, b_2, d_1, and e_1 were obtained with the help of symbolic math of Matlab [14], but are not listed here because they are too complicated. Instead, a Matlab program to obtain the coefficients is listed in Appendix A to help readers generate the solutions. Examining the solutions, it was found that the coefficients are proportional to F_0, which was expected. It was also found that a_1, b_1, a_2, and b_2 are independent of r, and d_1 and e_1 are inversely proportional to r. It means that the magnitudes of the frequency components of x are independent of r and the magnitude of the frequency component of θ is inversely proportional to r.

The above observation was confirmed with simulation results. For the previous simulation, the magnitudes of the frequency component of x at ω_1 and $2\omega_2-\omega_1$ were 0.01029 and 0.005334 m, respectively, and that of θ at $\omega_1-\omega_2$ was 0.1855 rad. When the value of r was doubled with the other parameters fixed (J was increased by four times to make the natural frequency of the rotational mode, ω_{nr}, unchanged), the frequency components of x were 0.01029 and 0.005352 m, and that of θ was 0.09267 rad. That is, the frequency components of x were unchanged and that of θ was halved. This result confirms the above observation.

4. Conclusions

The motion of a system composed of a plate, constant springs and varying dampers was considered when the system is subject to harmonic force. Letting the frequencies of harmonic force and damper variation f_1 and f_2, respectively, the displacement at the center of the plate has frequency components at $2f_2-f_1$, f_2-f_{nr}, f_2+f_{nr}, f_1, $4f_2-f_1$, $2f_2+f_1$, and so on, where f_{nr} represents the natural frequency of the rotational mode of the plate. Among these frequency components, the strongest one is the component at frequency f_1, and its magnitude is almost independent of f_2. The next strongest component is that at $2f_2-f_1$. The angular displacement of the plate has frequency components at f_{nr}, f_1-f_2, $3f_2-f_1$, f_1+f_2, $5f_2-f_1$, $3f_2+f_1$, and so on. Among these frequency components, two components at f_1-f_2 and f_{nr} are prominent. If these two frequencies coincide, the plate oscillates with almost a single frequency and a large amplitude.

The frequency components present in a homogeneous case without external excitation occur also in the steady-state response of a case with external excitation.

The existence of the frequency components of the vertical and rotational displacements was proved analytically. For a special case where f_1-f_2 is equal to f_{nr}, approximate expressions for the vertical and rotational displacements were derived. They show that the magnitudes of the frequency components of the vertical displacement are independent of r and the magnitude of the frequency component of the rotational displacement is inversely proportional to r, where r represents the distance from the center of the plate to a spring and a damper.

Acknowledgments

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Appendix A. Matlab program to obtain the coefficients of frequency components

```
syms m k c r F0 w1 w2 a1 a2 b1 b2 d1 e1 pi
A = [2*k—m*w1^2 -c*w1 0 0 2/pi*c*r*(w1—w2) 0; ...
  c*w1 2*k—m*w1^2 0 0 0 2/pi*c*r*(w1-w2); ...
  0 0 2*k—m*(2*w2—w1)^2 —c*(2*w2—w1) 2/pi*c*r*(w1—w2) 0; ...
  0 0 c*(2*w2—w1) 2*k—m*(2*w2—w1)^2 0 —2/pi*c*r*(w1—w2); ...
  —2/pi*c*r*w1 0 2/pi*c*r*(2*w2—w1) 0 0 —c*r^2*(w1—w2); ...
  0—2/pi*c*r*w1 0 -2/pi*c*r*(2*w2-w1) c*r^2*(w1-w2) 0];
b = [F0; 0; 0; 0; 0; 0];
z = A\b
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